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FORCED CONVECTION IN PARALLEL AND TAPERED PASSAGES: THE IMPORTANCE OF THE HEAT FLUX BOUNDARY CONDITION

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Abstract—It has been established previously that, for heat transfer to a constant property fluid in a long pipe, an exponential wall heat flux ($q \propto e^{\gamma x}$) leads to a fully-developed Nusselt number. Attention is drawn to the rapid decrease of Nu as the parameter γ is made increasingly negative. For wedge flow there is a corresponding family of thermal boundary conditions for which Nu is constant in the flow direction. If the familiar uniform wall heat flux were to be adopted without reflection for experiments with converging passages, significant reductions in Nu could well be observed and might be attributed to other effects in the accelerated flow.

NOMENCLATURE

A,	constant;
A_n, C_n,	coefficients;
d,	passage diameter or equivalent diameter;
K,	acceleration parameter;
q,	heat flux per unit area at the passage boundary;
r,	radial coordinate;
u,	velocity;
$\bar{u}$,	mean velocity across the passage;
u',	fluctuating component of velocity;
x,	coordinate in flow direction;
X,	x/d;
y,	coordinate across the passage;
Y,	y/d;
Nu,	Nusselt number;
Pr,	Prandtl number;
Re,	Reynolds number;
St,	Stanton number;
α,	thermal diffusivity;
β_n^2,	eigenvalues;
γ,	dimensionless parameter;
ε,	eddy diffusivity for heat;
θ,	temperature;
θ',	fluctuating component of temperature;
ϕ,	angular coordinate;
Φ,	twice the wedge angle;
$\varkappa$,	dimensionless coordinate in wedge flow;
Ψ,	ϕ/Φ;
Ψ_n,	eigenfunctions;
b,	suffix denoting bulk value.

INTRODUCTION

WE BEGIN by recapitulating briefly a previous treatment of thermal boundary condition for a parallel passage. We then apply the method of superposition to the same problem, mainly to extend (in Fig. 1) the available numerical results. The Nusselt numbers for exponentially decreasing wall heat fluxes are seen to be particularly significant and are discussed. The thermal boundary condition analysis is then extended to the case of flow in an infinite wedge, and links are established between the parallel flow and wedge flow cases. Finally the preceding analytical development is used to estimate the contribution, by thermal boundary condition alone, to the reported heat transfer results from an experiment employing a converging annulus.

PARALLEL PASSAGES

For the fully-developed flow of a constant property fluid in a long pipe, the local heat transfer coefficient is dependent upon the thermal boundary condition. If the uniform wall heat flux is taken as the standard case, then by comparison the heat transfer coefficient is enhanced where the wall heat flux is increasing in the flow direction and reduced where the heat flux is decreasing. It has been further shown by Hall, Jackson and Price [1] that the heat transfer coefficient becomes independent of the axial position X whenever the wall heat flux over a long passage varies as $e^{\gamma X}$, the fully-developed heat-transfer coefficient being dependent upon the parameter γ. The uniform heat flux case is obtained as $\gamma \to 0$, and another, negative, value of γ corresponds to the constant wall temperature boundary condition, with a heat transfer coefficient somewhat lower than the uniform heat flux case.

For later reference it is useful to recapitulate the main points in the development by Hall, Jackson and Price. Consider a passage formed by two infinite parallel planes at $Y = \pm\frac{1}{4}$. For constant physical properties the governing equation is

$$\frac{\partial}{\partial Y}\left[\left(1+\frac{\varepsilon}{\alpha}\right)\frac{\partial \theta}{\partial Y}\right] = \frac{u}{\bar{u}} Re\, Pr \frac{\partial \theta}{\partial X}. \quad (1)$$

Following the procedure for the separation of variables we write

$$\theta = f(Y)\, g(X)$$

and, after substitution into (1) and rearrangement, set the two expressions in X and Y respectively equal to a constant, γ. Equation (1) is thus replaced by two ordinary differential equations:

$$\frac{dg}{dX} - \gamma g = 0 \quad (2)$$

$$\frac{d}{dY}\left[\left(1+\frac{\varepsilon}{\alpha}\right)\frac{df}{dY}\right] - Re\, Pr \frac{u}{\bar{u}} \gamma f = 0. \quad (3)$$

A solution of (2) is $e^{\gamma X}$. Let a solution of (3) be $f(Y)$, containing also the parameter γ. Ignoring

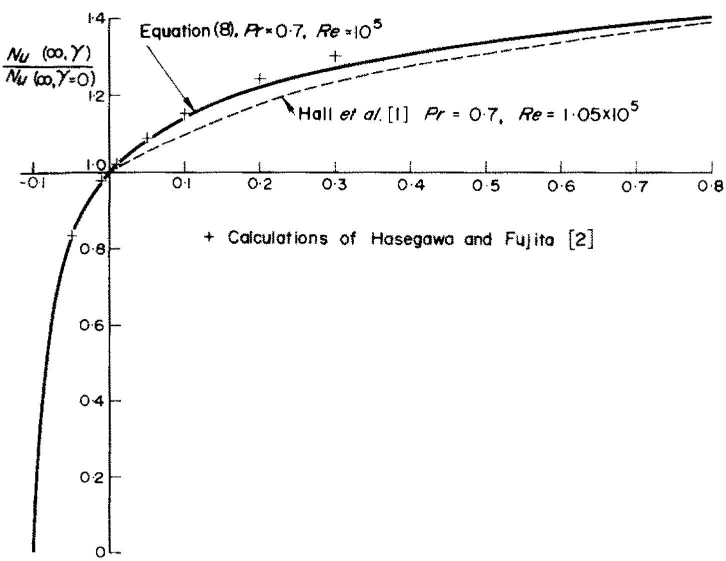

FIG. 1. Influence of exponential wall heat flux on fully-developed Nusselt number.

an additive constant, a solution of (1) is

$$\theta = Af(Y)\,e^{\gamma X}. \tag{4}$$

This represents a temperature distribution invariant in form $f(Y)$ across the passage, but scaled up in proportion to $e^{\gamma X}$. As the temperature gradient at the wall, and therefore the heat flux, is also proportional to $e^{\gamma X}$, the heat transfer coefficient is independent of X.

heat flux $q(x)$ ($x \geqslant 0$) in the form

$$\theta(x, r) = q(0)\,\phi(x, r) + \int_0^x \frac{dq(\xi)}{d\xi}\,\phi(x - \xi)\,d\xi. \tag{6}$$

We are interested in the exponential wall heat flux $q(x) = e^{\gamma x/d}$. It follows that the Nusselt number for any x and a given γ, normalized by the fully-developed $Nu(x = \infty)$ for a uniform heat flux ($\gamma = 0$) is given by

$$\frac{Nu(x, \gamma)}{Nu(\infty, \gamma = 0)} = 1/1 + \sum_1^\infty A_n\left\{\frac{\gamma}{4\beta_n^2/Re + \gamma} + \left[1 - \frac{\gamma}{4\beta_n^2/Re + \gamma}\right]\exp\left[-\left(\frac{4\beta_n^2}{Re} + \gamma\right)\frac{x}{d}\right]\right\}. \tag{7}$$

By solving equation (3) numerically for positive values of γ, Hall, Jackson and Price were able to show the influence of the exponentially increasing wall heat flux on the fully-developed Nusselt number and their curve is reproduced in Fig. 1; some calculated results of Hasegawa and Fujita [2] also are shown. An alternative approach, which leads to the extension of the data in Fig. 1, is to construct such a curve by starting with a solution for a uniform heat flux for $x > 0$ following a step change at $x = 0$, and employing superposition to develop a solution for the exponential heat flux.

A solution for the temperature distribution in a circular pipe with fully-developed flow and unit heat flux starting at $x = 0$ has been given by Sparrow, Hallman and Siegel [3]:

$$\phi(x, r) = \phi_b + [\phi(\infty, r) - \phi_b] + \sum_1^\infty C_n \Psi_n(r)\exp\left[-\frac{4\beta_n^2}{Re}\frac{x}{d}\right] \tag{5}$$

where, on the right-hand side,

(i) the first term is the bulk temperature rise;
(ii) the second term is the fully-developed radial temperature distribution;
(iii) the third term is the modification of the temperature distribution in the entry region.

The superposition principle gives the temperature distribution $\theta(x, r)$ arising from a wall

Provided $4\beta_1^2/Re + \gamma > 0$, this expression becomes independent of x at very large x, confirming that the exponential wall heat flux leads to a fully-developed Nusselt number varying with γ according to

$$\frac{Nu(\infty, \gamma)}{Nu(\infty, \gamma = 0)} = \frac{1}{1 + \sum_1^\infty \frac{\gamma}{4\beta_n^2/Re + \gamma}A_n}. \tag{8}$$

Sparrow, Hallman and Siegel have tabulated the first five values of β_n^2 and A_n for a number of cases. Equation (8) has been evaluated for $Re = 10^5$, $Pr = 0{\cdot}7$ and a range of values of γ, both positive and negative, and the results are shown in Fig. 1. As the three sets of calculations shown all relate to circular pipes the discrepancies presumably arise from differences in eddy diffusivity data and numerical integration procedures.

It is noted that for these values of Re and Pr, the constant temperature boundary condition is equivalent to $\gamma = -0{\cdot}011$ (i.e. $-4St$), and the curve shows the corresponding Nu to be 2 per cent below the uniform heat flux case. The rapid fall in Nu with further reduction of γ is apparent. For the case plotted in the figure the value of $4\beta_1^2/Re$ as given by Sparrow *et al.* is 0·102. As γ approaches $-0{\cdot}102$ from above, the expression in equation (8) behaves as

$$\frac{Nu(\infty, \gamma)}{Nu(\infty, \gamma = 0)} \approx \frac{1}{1 + \frac{A_1\gamma}{4\beta_1^2/Re + \gamma}} \tag{9}$$

(where $A_1 < 0$) and tends to zero as $\gamma \to -0{\cdot}102$. For $\gamma < -0{\cdot}102$ the exponential terms in the series (7) do not all vanish at large x and the expression does not lead to a fully-developed Nusselt number.

We pursue this point by reconsidering the differential equation (3) for the form of the temperature distribution $f(Y)$ across the passage:

$$\frac{\mathrm{d}}{\mathrm{d}Y}\left[\left(1+\frac{\varepsilon}{\alpha}\right)\frac{\mathrm{d}f}{\mathrm{d}Y}\right] - Re\,Pr\frac{u}{\bar{u}}\gamma f = 0.$$

The numbers $-4\beta_n^2/Re$ are the eigenvalues of this equation when it has the boundary condition $\mathrm{d}f/\mathrm{d}Y = 0$ at $Y = \pm\frac{1}{4}$ i.e. the values of γ for which, and only for which, f has zero slope at the passage boundaries. As γ, decreasing, passes through the value $-4\beta_1^2/Re$, the form of the distribution f changes as illustrated in Fig. 2. In all cases the temperature distribution

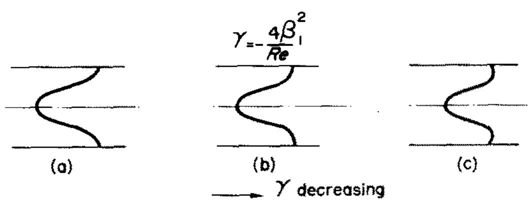

FIG. 2. Variation of temperature distribution for γ in vicinity of first eigenvalue.

decays as $e^{\gamma x}$. So does the wall heat flux with the proviso that when $\gamma = -4\beta_1^2/Re$ the coefficient of $e^{\gamma x}$ is zero. For $\gamma < -4\beta_1^2/Re$, the main characteristic of the temperature profile is internal redistribution of energy rather than heat flow from the wall, and it would appear, from the breakdown of the approach via superposition for $\gamma < -4\beta_1^2/Re$, that it is impossible to develop these temperature distributions by starting with a fluid of uniform temperature at the passage entry and applying a wall heat flux distribution.

TAPERED PASSAGES

Consider the wedge flow in the r-direction between the infinite planes depicted in Fig. 3. If the wedge is long in the r-direction the flow becomes dependent on just two parameters Re

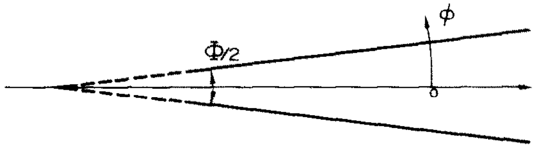

FIG. 3. Wedge flow definitions.

and $(d/\bar{u})(\mathrm{d}\bar{u}/\mathrm{d}r)$ (which is equal to $-\Phi$); (a common alternative form of the acceleration parameter is

$$\frac{1}{Re}\frac{d}{\bar{u}}\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} = \frac{\nu}{\bar{u}^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}r} \equiv K).$$

Both parameters are independent of r, hence the flow becomes fully-developed. Heat transfer depends additionally upon Pr and the thermal boundary condition, the investigation of which may be undertaken in the same manner as for parallel flow. The counterpart of equation (1) is

$$\frac{\partial}{\partial\phi}\left[\left(\alpha+\varepsilon\right)\frac{1}{r}\frac{\partial\theta}{\partial\phi}\right] = \frac{\partial}{\partial r}(u\theta r). \qquad (10)$$

When the flow is fully-developed, lines of constant ϕ are streamlines, and the product ur is constant along a streamline. It is also readily seen from similarity arguments that ε is independent of r: ε results from the product of a mixing length (proportional to r) and a velocity (inversely proportional to r): an alternative argument is that $\varepsilon/r\;\partial\theta/\partial\phi$ is related to $\overline{u'\theta'}$ where u' behaves as $1/r$ and θ' as $\partial\theta/\partial\phi$, hence ε is independent of r. Equation (11) therefore reduces to the following form

$$\frac{1}{\Phi}\frac{\partial}{\partial\Psi}\left[\left(1+\frac{\varepsilon}{\alpha}\right)\frac{\partial\theta}{\partial\Psi}\right] = \frac{u}{\bar{u}}Re\,Pr\,r\frac{\partial\theta}{\partial r}. \qquad (11)$$

If we again separate the variables by writing $\theta = F(\Psi)\,G(r)$ we obtain the two equations

$$\frac{\mathrm{d}G}{\mathrm{d}r} - \frac{\gamma G}{r\Phi} = 0 \qquad (12)$$

$$\frac{\mathrm{d}}{\mathrm{d}\Psi}\left[\left(1+\frac{\varepsilon}{\alpha}\right)\frac{\mathrm{d}F}{\mathrm{d}\Psi}\right] - Re\, Pr\frac{u}{\bar{u}}\gamma F = 0. \quad (13)$$

A solution of (12) is

$$G = r^{\gamma/\Phi}. \quad (14)$$

For flow in the negative r direction the corresponding result is

$$G = r^{-\gamma/\Phi} \quad (14a)$$

in effect the same result but with a negative value of Φ. The family of temperature profiles, corresponding to (4) for a parallel passage, is

$$\theta = AF(\Psi)\, r^{\gamma/\Phi}. \quad (15)$$

The wall heat flux $1/r\ \partial\theta/\partial\Psi$ at $\Psi = \frac{1}{4}$ therefore varies as

$$\dot{q}_w \propto r^{\gamma/\Phi - 1} \quad (16)$$

and the Nusselt number follows:

$$Nu \propto \frac{\dot{q}_w(r\Phi)}{\theta} = \text{constant}. \quad (17)$$

It is concluded that the variation of wall heat flux (16) is, for a wedge flow, the counterpart of the exponential variation for a parallel flow, in leading to a fully-developed Nu for each value of γ.

An alternative approach to the same result would be to transform equation (12) by introducing a new variable $\varkappa$, the result of making the distance r dimensionless by means of the local equivalent diameter i.e.

$$\mathrm{d}\varkappa = \frac{1}{r\Phi}\,\mathrm{d}r.$$

Equation (13) becomes

$$\mathrm{d}G/\mathrm{d}\varkappa - \gamma G = 0 \quad (12a)$$

with solution $G = \mathrm{e}^{\gamma\varkappa} = r^{\gamma/\Phi}$.

It is seen that both the parallel and tapered passages give rise to the exponential function provided that the unit of distance in the flow direction is the local equivalent diameter.

Nusselt numbers for wedge flow are not deducible from those for parallel flow without substantial simplifying assumptions. Suppose in equation (13) we make these simplifications by ignoring the influence of acceleration on the velocity distribution $u/\bar{u}$ and on the eddy diffusivity of heat ε. Then equations (3) and (13) are identical except for the changes of symbols Y to Ψ and f to F, and are both to be solved for $-\frac{1}{4} < Y(\text{or } \Psi) < \frac{1}{4}$. The solutions f and F are identical. It follows that

Nu (parallel flow) with wall heat flux $\mathrm{e}^{\gamma x}$

and

Nu (wedge flow) with wall heat flux $r^{\gamma/\Phi - 1}$

are identical (subject to the important approximations regarding the similarity of $u/\bar{u}$ and ε). Suppose $\gamma = 0$, a pair of corresponding thermal boundary conditions are that the wall heat flux is constant for parallel flow and varies as $1/r$ for wedge flow. In both cases the heat flux and the heat transfer coefficient vary in the same manner. They are both examples of constant temperature difference and it is intuitively obvious that these are the corresponding boundary conditions for equal Nu.

Heat transfer in accelerating flow is an active field of research and the constant heat flux boundary condition, imposed electrically, is a convenient experimental arrangement. It implies $\gamma - \Phi = 0$. For converging wedge flow, Φ is negative and $\gamma = -|\Phi|$. An included angle of less than 3°, in conjunction with a uniform wall heat flux, represents a γ of $-0{\cdot}1$, which, in the parallel flow case examined, Fig. 1, was sufficient to reduce the fully-developed Nusselt number to zero. It is apparent that very severe reductions in Nu are likely to be observed in converging passage flow experiments unless particular thought is given to the thermal boundary condition.

Experiments with converging annular flow passages have been reported by Fujita and Hasegawa [4]. The core was cylindrical and formed by a thin tube which was electrical resistance heated to give an approximately uniform heat flux. The outer boundary of the annulus was an adiabatic surface, the diameter varying along the length to provide the converging passage. It was reported that the local Stanton number in the converging annulus fell below the full-developed value for a parallel annulus with the same dimensions, to an extent which increased with the acceleration parameter K, and the effect was attributed to the reduced turbulence production in the accelerated boundary layer, a phenomenon which is receiving widespread attention at the present time. It is of interest to try and assess how much of the observed decline in St might have been caused by the thermal boundary condition effect discussed in the present paper. The analytical procedure involving separation of variables cannot be developed for a converging annulus owing to the inherent variation of geometry (radius ratio) in the axial direction. As an approximation we shall perform calculations for a parallel annulus of suitable radius ratio and relate the results to a convergent annulus by using the connections previously established between a flow between parallel planes and a wedge flow.

In Fig. 4 the experimental curves for two rates of change of flow area have been taken from Fig. 16 of [4]. The distance scale has been transformed and made dimensionless by means of the local equivalent diameter. In the experiments Re varied along the converging passage by a factor 1·4. In an attempt to eliminate this effect the experimental St have been multiplied by (local $Re)^{0\cdot 2}$. In these particular experiments there was a long parallel thermal entry section upstream of the converging portion, and the curves have been normalized with respect to the fully-developed St upstream. In this form the curves show the combined effects of the influence of acceleration on $u/\bar{u}$ and ε, the thermal boundary condition, and the axial variation of the annulus radius ratio from 2·56 to 1·5.

The other two curves in Fig. 4 indicate to a first approximation the effect of thermal boundary condition alone. They were obtained from calculations for a parallel annulus, of radius ratio 2·0, with heat flow from the core tube only as in the experiment, with a long uniform heat flux entry section followed by an exponential wall heat flux $e^{\gamma x/d_e}$ from $x = 0$. The two values of γ, corresponding to the two converging annuli with uniform heat flux, were $-0{\cdot}0401$ and $-0{\cdot}0576$. The calculations were made by the numerical method of Patankar and Spalding [5]. It is emphasised that this was not an attempt to predict Stanton number distributions for com-

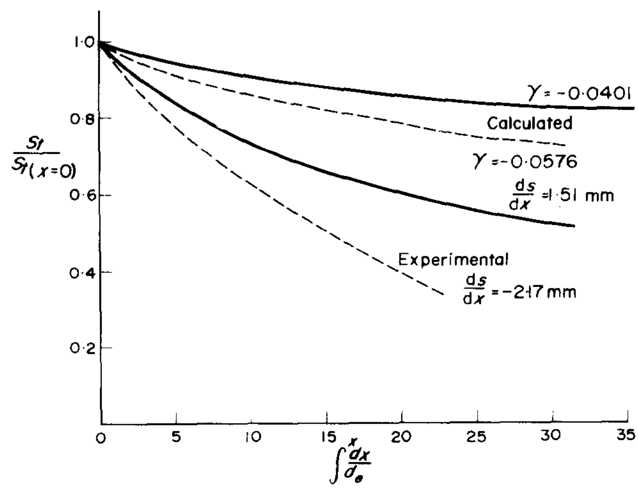

FIG. 4. Approximate influence of thermal boundary conditions in experiments of [4].

parison with the experimental curves, but simply to estimate the effect of the experimental thermal boundary condition. By the choice of a parallel annulus, which could then be related to the convergent passage by use of the distance coordinate $\int dx/d_e$, other effects of acceleration such as velocity distribution and eddy diffusivity changes, were deliberately excluded.

CONCLUSIONS

For parallel passages the rate of decrease of Nu as the parameter γ, in the exponential heat flux boundary condition, becomes negative, is noteworthy.

For wedge flow there is again a family of thermal boundary conditions for which fully-developed temperature distributions and Nusselt numbers result. If the unit of distance in the flow direction is taken as the local equivalent diameter, the temperature profiles are scaled by an exponential function of the distance for both parallel and wedge flow.

If differences in velocity distribution and eddy diffusivity are ignored, there is a direct correspondence between the uniform wall heat flux case in parallel flow, and heat flux inversely proportional to gap in wedge flow. They are both conditions of constant temperature difference.

For accelerating flow in a wedge, the constant heat flux boundary condition leads to a decreasing temperature difference and hence reduced Nu. Caution is therefore needed in adopting this convenient boundary condition for such experiments. In the experiments considered it would appear to have been a significant effect. For rather greater rates of convergence it could become dominant.

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CONVECTION FORCÉE DANS DES PASSAGES PARALLÈLES OU CONVERGENTS: IMPORTANCE DE LA CONDITION AUX LIMITES DE FLUX THERMIQUE

Résumé—On a antérieurement établi que, pour un transfert thermique à un fluide à propriété constante dans un long tube, un flux thermique pariétal exponentiel ($q \propto e^{\gamma x}$) conduit à un nombre de Nusselt entièrement développé. On porte l'attention sur la décroissance rapide de Nu lorsque le paramètre γ est de plus en plus négatif. Pour un écoulement autour d'un coin il y a une famille correspondante de conditions limites thermiques pour lesquelles Nu est constant dans la direction de l'écoulement. En adoptant sans méfiance l'hypothèse d'un flux thermique uniforme à la paroi pour des expériences relatives à des passages convergents, des réductions significatives de Nu pourraient être observées et attribuées à d'autres effets dans l'écoulement accéléré.

ERZWUNGENE KONVEKTION IN KANÄLEN MIT PARALLELEN UND KONISCHEN WÄNDEN: DIE BEDEUTUNG DER RANDBEDINGUNG FÜR DEN WÄRMESTROM

Zusammenfassung—Es wurde früher festgestellt, dass bei Wärmeübertragung in einem langen Rohr an ein Fluid mit konstanten Stoffwerten ein exponentielles Verhalten des Wärmestroms durch die Wand ($q \sim e^{\gamma x}$) zu einer konstanten Nusselt-Zahl führt. Bemerkenswert ist die schnelle Abnahme der Nusselt-Zahl, wenn der Parameter γ zunehmend negativ wird. Bei Keilströmungen gibt es eine entsprechende Reihe von Randbedingungen, für die Nu in Strömungsrichtung konstant bleibt. Wenn der bekannte gleichförmige Wärmestrom durch die Wand auf konvergierende Kanäle angepasst würde, ohne Berücksichtigung der Experimente, wäre eine deutliche Abnahme der Nusseltzahl zu beobachten, die dann anderen Effekten in einer beschleunigten Strömung zugeordnet werden könnte.

СВОБОДНАЯ КОНВЕКЦИЯ В ПАРАЛЛЕЛЬНЫХ И КОНИЧЕСКИХ КАНАЛАХ. РОЛЬ ГРАНИЧНОГО УСЛОВИЯ ТЕПЛОВОГО ПОТОКА

Аннотация—Ранее было установлено, что в случае переноса тепла к жидкости с постоянными свойствами в длинной трубе экспоненциальный тепловой поток на стенке ($g\alpha e^{\gamma x}$) приводит к полностью развитому числу Нуссельта. Обращается внимание на быстрое снижение числа Нуссельта, когда параметр γ становится возрастающе отрицательным. Для обтекания клина имеется соответствующее семейство тепловых пограничных условий, для которых число Нуссельта постоянно в направлении течения. Если для экспериментов в сужающихся каналах использовать известный равномерный тепловой поток на стенке, отчетливо обнаружилось бы значительное снижение числа Нуссельта, которое можно было бы объяснить другими факторами в ускоряющемся потоке.